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Bipartite Matching and Van der Waerden Conjecture

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In this paper I show that the free energy F and the cost C associated to a bipartite matching problem can be explicitly estimated in term of the solution of a suitable system of equations (cavity equations in the following). The proof of these results relies on a well known result in combinatorics: the Van der Waerden conjecture (Egorychev–Falikman Theorem). Cavity equations, derived by a mean field argument by Mèzard and Parisi, can be considered as a smoothed form of the dual formulation for the bipartite matching problem. Moreover cavity equation are the Euler–Lagrange equations of a convex functional G parameterized by the temperature T. In term of their unique solution it is possible to define a free-energy-like function of the temperature $g(T)$. g is a strictly decreasing concave function of T and $C = g(0)$. The convexity of G allows to define an explicit algorithm to find the solution of the cavity equations at a given temperature T. Moreover, once the solution of the cavity equations at a given temperature T is known, the properties of g allow to find exact estimates from below and from above of the cost C.

KEY WORDS: Bipartite matching; cavity equations; statistical mechanics.

1. INTRODUCTION

A bipartite matching problem is defined as follows: given a $N \times N$ square matrix $A_{i,j}$ with nonnegative elements one is interested to minimize

$$\sum_i A_{i,\Pi_i} \tag{1.1}$$

on the set of all the permutation $\Pi = \Pi_1,\dots, \Pi_N$ of the first N integers.

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This problem has many applications.

For example there are N jobs, $i=1,2,\ldots,N$ and N workers $j=1,2,\ldots,N$. The cost of the assignment of the job i to the worker j is $A_{i,j}$, and one has to assign to any worker a job in such a way that the total cost is minimum.

Alternatively, given $N+N$ points in a square: x_i: $i=1,\ldots,N$, and y_j: $j=1,\ldots,N$, one is interested to couple any point x_i to another point y_j in such a way that the sum of the pair distances is minimized. In this case we obtain a bipartite problem (generally called Euclidean Matching) and $A_{i,j}=|x_i-x_j|^{\alpha}$, for some $\alpha \geqslant 1$.

The bipartite matching is a widely studied problem in graph theory and fast algorithms have been obtained to solve it (see, e.g., ref. 1). These algorithms are based on the dual formulation for this problem which is briefly recalled here. This is a classical result in combinatorics (see, e.g., ref. 2).

Let p_i: $i=1,2,\ldots,N$, and q_j: $j=1,2,\ldots,N$, the $2N$ be dual variables. Then $C(A)$ is the maximum of

$$\sum_{i=1}^{N} p_i+\sum_{j=1}^{N} q_j \tag{1.2}$$

when the variables p and q satisfy the N^2 constraints $p_i+q_j \leqslant A_{i,j}$.

In a series of paper Mèzard and Parisi consider the random bipartite matching problem. In this case the entries of the matrix $A_{i,j}$ are i.i.d. random variables extracted from the uniform distribution in $[0,1]$. Here the crucial point is the determination of the average value $\langle C\rangle$ of $C(A)$ on the ensemble of matricies A.

For $\beta \geqslant 0$ they define the partition function

$$Z_{N,\beta}=\sum_{\Pi} e^{-\beta \sum_i A_{i,\Pi_i}} \tag{1.3}$$

and

$$F_{N,\beta}=-\frac{1}{\beta}\log Z_{N,\beta} \tag{1.4}$$

Obviously

$$\lim_{\beta\to\infty} F_{N,\beta}=C(A) \tag{1.5}$$

Then, using the replica approach,[3] they are able to find an analytical expression for $\langle C \rangle$ in the mean field limit $N \to \infty$. More precisely they find $\langle C \rangle \to \pi^2/6$ as $N \to \infty$.

In ref. 4 they have confirmed this result using a cavity approach, while in ref. 5 their result has been numerically tested. Rigorous constructive bounds of $\langle C \rangle$ have been given in ref. 2, while D. J. Aldous,[6] with an ingenious construction, has definitively proven that $\langle C \rangle \to \pi^2/6$ as $N \to \infty$.

Moreover in ref. 7 Parisi conjectured that in the case in which the entriees of the matrix are extracted from the exponential distribution $e^{-x}\,dx$, the average value is $C_N = \sum_{k=1}^{N} 1/k^2$ for any N !

In this paper I will show that thanks to the Van der Waerden conjecture (proved by Egorychev[8] and by Falikman[9]) it is possible to estimate $F_{N,\beta}$ from above and from below in terms of the solutions of the cavity equations proposed by Mèzard and Parisi in ref. 4.

Moreover these equations are satisfied on the unique minimum $\bar{G}_{N,\beta}$, of a convex function of $2N$ real variables $G_{N,\beta}$ parametrized by β.

The cost of the bipartite matching problem can be recovered from the solution when $T = 0$, where $T = \frac{1}{\beta}$ is the temperature.

More precisely defining $g(T) = NT - T\bar{G}_{N,\frac{1}{T}}$, it turns out that $C = g(0)$.

Furthermore g can be considered as a free energy. In particular g is a strictly decreasing concave function of the temperature T.

Finally, in this way, it is possible to recover the usual dual formulation for this problem as $T \to 0$.

From my perspective the most relevant result here is the definition of the one parameter (β) family of relaxed problems and their "thermodynamical" properties.

By using this construction, it is possible to define numerical algorithms to solve the problem.

In fact in order to obtain $g(T)$ one has to find the minimum of a convex function of $2N$ real variables. Moreover the monotonicity and concavity of $g(T)$ allows us, once that the solution at a certain T is known, to estimate from above and from below the cost $g(0)$.

The algorithms I propose here does not seem particularly good and probably their performance is comparable to that of the simplex method. Anyway it may be that proceeding along this line it is possible to obtain better results.

The outline of the paper is as follows. In Section 2 I define the cavity equation and their relation with the partition function. In Section 3 I define the function g, and I discuss its properties. Finally in Section 4 an algorithm to solve the matching problem, based on this formalism, is defined and numerically tested.

2. CAVITY EQUATIONS AND VAN DER WAERDEN CONJECTURE

The partition function for a bipartite matching problem defined in (1.3) can be written as the permanent of the $N \times N$ matrix M defined by $M_{i,j} = e^{-\beta A_{i,j}}$.

In fact the permanent $\mathrm{Per}(M)$ of a matrix M is defined as the sum on all the permutations, $\Pi = \Pi_1, \ldots, \Pi_N$, of the first N integers Π of the product M_{i,Π_i}.

Notice that the permanent is defined as the determinant, except for the signs of the permutations, omitted in the permanent.

The Van der Waerden conjecture states that if a $N \times N$ matrix with nonnegative entries is such that the sum of its elements in any row and in any column is 1 then its permanent can be estimated from below by $\frac{N!}{N^N}$, while the estimate from above is obviously 1.

By multiplying the ith row of the matrix M by s_i and its jth column by t_j (for any i and for any j) one finds the matrix

$$L_{i,j} = M_{i,j} s_i t_j = e^{-\beta A_{i,j}} s_i t_j \tag{2.2}$$

If the quantities s_i and t_j satisfy the equations

$$\begin{aligned} s_i &= \frac{1}{\sum_j e^{-\beta A_{i,j}} t_j} : \quad i = 1, 2, \ldots, N \\ t_j &= \frac{1}{\sum_i e^{-\beta A_{i,j}} s_i} : \quad i = 1, 2, \ldots, N \end{aligned} \tag{2.1}$$

then the matrix L satisfies the hypothesis of the Van der Waerden conjecture. Therefore $\mathrm{Per}(L)$ can be estimated as said above.

Equations (2.1) are exactly the cavity equations proposed by Mèzard and Parisi in ref. 4, on the basis of a mean field argument.

From

$$\mathrm{Per}(L) = \mathrm{Per}(M) \prod_{i=1}^{N} s_i \prod_{j=1}^{N} t_j \tag{2.3}$$

one obtains

$$\frac{N!}{N^N} \frac{1}{\prod_{i=1}^N s_i \prod_{j=1}^N t_j} \leqslant \mathrm{Per}(M) = Z \leqslant \frac{1}{\prod_{i=1}^N s_i \prod_{j=1}^N t_j} \tag{2.4}$$

Taking the logarithm of both members in (2.4) and dividing by β one finds

$$\frac{1}{\beta}\log\left(\prod_{i=1}^{N} s_i \prod_{j=1}^{N} t_j\right) \leqslant F_{N,\beta} \leqslant \frac{1}{\beta}\log\frac{N^N}{N!} + \frac{1}{\beta}\log\left(\prod_{i=1}^{N} s_i \prod_{j=1}^{N} t_j\right) \tag{2.5}$$

Notice that $\frac{N!}{N^N}$ does not depend on β, therefore in the limit $\beta \to \infty$ its contribution to the estimate is negligible.

More precisely, by recalling that $\frac{N!}{N^N} \geqslant e^{-N}$, one gets that $F_{N,\beta}$ is estimated in terms of the solution of the cavity equations with an error bounded by $\frac{N}{\beta}$.

Finally, from $e^{-\beta C} \leqslant Z_{N,\beta} \leqslant N!\,e^{-\beta C}$ and Eq. (2.4) one gets

$$\frac{1}{\beta}\log\left(\prod_{i=1}^{N} s_i \prod_{j=1}^{N} t_j\right) \leqslant C \leqslant \frac{N\log N}{\beta} + \frac{1}{\beta}\log\left(\prod_{i=1}^{N} s_i \prod_{j=1}^{N} t_j\right) \tag{2.6}$$

In particular

$$C = \lim_{\beta\to\infty} \frac{1}{\beta}\log\left(\prod_{i=1}^{N} s_i \prod_{j=1}^{N} t_j\right) \tag{2.7}$$

3. A FREE-ENERGY-LIKE FUNCTIONAL

In order to study the cavity equations it is convenient to introduce the variables σ_i: $i = 1, 2,\dots, N$, and τ_j: $j = 1, 2,\dots, N$, defined by

$$\begin{aligned} s_i &= e^{\beta\sigma_i}: \quad i = 1,\dots, N \\ t_j &= e^{\beta\tau_j}: \quad j = 1,\dots, N \end{aligned} \tag{3.1}$$

The cavity equations (2.1) become

$$\begin{aligned} e^{-\beta\sigma_i} &= \sum_{j=1,N} M_{i,j} e^{\beta\tau_j}, \\ e^{-\beta\tau_j} &= \sum_{i=1,N} M_{i,j} e^{\beta\sigma_i} \end{aligned} \tag{3.2}$$

It is easy to check that the Eqs. (3.2) are satisfied on the stationary points of the following functional:

$$G_{N,\beta} = -\beta\left(\sum_{i=1}^{N} \sigma_i + \sum_{j=1}^{N} \tau_j\right) + \sum_{i,j} M_{i,j} e^{\beta(\sigma_i+\tau_j)} \tag{3.3}$$

Moreover one can notice that $G_{N,\beta}$ is invariant for $\sigma_i \to \sigma_i + a$ for any i, and $\tau_j \to \tau_j - a$ for any j.

It is therefore convenient to define $G_{N,\beta}$ on the set (it is a vector space)

$$I = \left\{ \sigma_1,..., \sigma_N, \tau_1,..., \tau_N : \sum_{i=1,N} \sigma_i - \sum_{j=1,N} \tau_i = 0 \right\} \tag{3.4}$$

On this set $G_{N,\beta}$ is convex.

Notice that the first addend of $G_{N,\beta}$,

$$G^{(1)}_{N,\beta} = -\beta \left(\sum_{i=1}^{N} \sigma_i + \sum_{j=1}^{N} \tau_j \right) \tag{3.5}$$

is linear while the second

$$G^{(2)}_{N,\beta} = \sum_{i,j} M_{i,j} e^{\beta(\sigma_i + \tau_j)} \tag{3.6}$$

is the sum of N^2 terms of the form $M_{i,j} e^{\beta(\sigma_i + \tau_j)}$ where $M_{i,j} > 0$. Since any of these terms is convex their sum is convex.

Finally the strict convexity of $G_{N,\beta}$, can be easily checked. This means that, for fixed β, there exists a unique solution of the cavity equations $\bar{\sigma}_1,..., \bar{\sigma}_N, \bar{\tau}_1,..., \bar{\tau}_N$.

Let us denote with $\bar{G}_{N,\beta}, \bar{G}^{(1)}_{N,\beta}, \bar{G}^{(2)}_{N,\beta}$ the values of $G_{N.\beta}, G^{(1)}_{N.\beta}, G^{(2)}_{N.\beta}$, on the solution, respectively, and let us notice that on the solution it holds $G^{(2)}_{N,\beta} = N$.

For our purposes it is convenient now to introduce the temperature $T = \frac{1}{\beta}$ and a new free energy $g(T)$.

Definition. Given $T > 0$ we define $g(T)$ as

$$g(T) = \frac{N}{\beta} - \bar{G}_{N,\beta}$$

where

$$\beta = \frac{1}{T}$$

Notice that it holds

$$g(T) = \sum_{i=1}^{N} \bar{\sigma}_i + \sum_{j=1}^{N} \bar{\tau}_j$$

For sake of simplicity I have omitted in g the explicit dependence upon N.

Now I will easily prove the following results.

Theorem 3.1.

(i) For $T>0$ $g(T)$ is a regular decreasing concave function, and its derivative, $-S$, is given by

$$\frac{dg}{dT} \equiv -S = -\sum_{i,j} \frac{(A_{i,j}-\bar{\sigma}_i-\bar{\tau}_j)}{T} \exp\left[-\frac{A_{i,j}-\bar{\sigma}_i-\bar{\tau}_j}{T}\right]$$

(ii) $\lim_{T\to 0} g(T) = C$;

(iii) $g(T) \leqslant g(0) = C \leqslant g(T) + T\,S(T)$.

Point (ii) is a consequence of Eq. (2.6).

(i) Since $G_{N,\beta}$ is a strictly convex function of $\sigma_1,\ldots,\sigma_N,\tau_1,\ldots,\tau_N$, for any $T>0$, then there exist the regular functions $\bar{\sigma}_1(T),\ldots,\bar{\sigma}_N(T)$, $\bar{\tau}_1(T),\ldots,\bar{\tau}_N(T)$.

One can therefore compute the derivative of g with respect to T, given by $-S$. S is positive because is the sum of positive terms. In fact, on the solution,

$$\bar{\sigma}_i = -\frac{1}{\beta}\log\left(\sum_{j=1}^{N} e^{-\beta(A_{i,j}-\bar{\tau}_j)}\right) \leqslant (A_{i,l}-\bar{\tau}_l) \qquad \text{for any } i, l$$

and therefore $A_{i,j}-\bar{\sigma}_i-\bar{\tau}_j \geqslant 0$ for any i, j.

It remains to prove that g is a concave function of T.

The concavity of g follows by its definition and the fact that $G_{N,\frac{1}{T}}$, is a convex function of the $2N+1$ variables $\sigma_1,\ldots,\sigma_N,\tau_1,\ldots,\tau_N,T$. In fact the term $-(\sum_{i=1}^N \sigma_i + \sum_{j=1}^N \tau_j)$ is linear, while the second term is the sum of N^2 terms of the form $e^{-(A_{i,j}-\sigma_i-\tau_j)/T}$ and any of these terms is a convex function of its three variables σ_i, τ_j, T. This can be explicitly checked by evaluating the 3×3 Hessian of this function.

Finally (iii) is a consequence of (i) and (ii). ∎

Remarks.

1. $g(T)$ can be interpreted as a free energy while $S=-\frac{dg}{dT}$ can be interpreted as an entropy. The analogy can be strenghtened by noticing that S can be written as

$$S = -\sum_{i,j} p_{i,j}\log p_{i,j}$$

where the variables $p_{i,j} = e^{-\beta(A_{i,j} - \bar{\sigma}_i - \bar{\tau}_j)}$: $i, j = 1, \ldots, N$, are nonnegative and satisfy $\sum_i p_{i,j} = 1$ for any j, $\sum_j p_{i,j} = 1$ for any i.

2. Let us notice that for $\beta \to \infty$ one easily recovers, from this formulation, the dual formulation defined with Eq. (1.2) by defining $q_i = \sigma_i$, $p_i = \tau_i$.

4. NUMERICAL ALGORITHMS

In this section I consider some possible algorithms to solve the bipartite matching problem based on the formulation introduced above.

First of all I introduce an algorithm, Algorithm A, to find the solution of the cavity equations at a given temperature T. In term of this solution it is possible to estimate from above and from below the cost C of the bipartite matching problem.

Then I consider an algorithm, Algorithm B which allows to find the solution of the bipartite matching problem starting close to it.

In terms of these two algorithm it is possible to define a simulated annealing algorithm to find the solution of the bipartite matching problem: Algorithm C.

Finally I discuss the numerical results I have obtained by using this algorithm for the random matching problem defined by Mèzard and Parisi, and I make some comments.

Let us now recall that the cavity equations are solved on the minimum of the convex functional $G_{N,\beta}$.

A possible way to minimize $G_{N,\beta}$ is given by the following algorithm.

Algorithm A. Starting with $\tau_{j=0}$ for any j, one iterates the following steps:

$$\text{(i)} \quad \sigma_i \to -\frac{1}{\beta} \log\left(\sum_{j=1}^{N} e^{-\beta(A_{i,j} - \tau_j)} \right) \tag{4.1}$$

$$\text{(ii)} \quad \tau_j \to -\frac{1}{\beta} \log\left(\sum_{1=1}^{N} e^{-\beta(A_{i,j} - \sigma_j)} \right) \tag{4.2}$$

In any of these operations $G_{N,\beta}$ decreases except when the solution of the problem has been already reached. In fact the minimum of $G_{N,\beta}$, when the variables $\tau_1, \ldots, \tau_N$ are fixed is given by (4.1), while the minimum of $G_{N,\beta}$, when the variables $\sigma_1, \ldots, \sigma_N$ are fixed, is given by (4.2). In this way one can find the solution $\bar{G}_{N,\beta}$.

Once the solution (a reasonable approximation of it) is found one can compute $g(T)$ and $S(T)$, see Theorem 3.1, obtaining the estimates

$$g(T) \leqslant C \leqslant g(T) + TS(T)$$

Obviously the complete solution can be recovered only when $T = 0$, but in this case the functional G is not convex anymore and then the previous algorithm cannot be used to find it.

Let us now recall an important consequence of the dual formulation of the bipartite matching problem.

Algorithm B. Let $\tau_1, \ldots, \tau_N$ be given. For any i let us consider a column j_i such that $A_{i,j} - \tau_j$ is minimum. If j_i defines a permutation, i.e., if do not exist two rows coupled to the same i, then one has found the optimal assignment which is given by $i - j_i$: $i = 1, \ldots, N$.

This can be easily proved. Indeed the matrix B, defined as $B_{i,j} = A_{i,j} - \sigma_i - \tau_j$, has nonnegative entries. Therefore its cost is nonnegative. But the assignment $i - j_i$ has a cost 0 therefore the matrix B has a cost 0, and the optimal assignment for B is $i - j_i$. Finally let us notice that the optimal assignment is invariant for the operation $A_{i,j} \to A_{i,j} - \sigma_i - \tau_j$.

Therefore a possible strategy to solve the bipartite matching problem is the following one. One solves the problem with Algorithm A for a sufficiently small temperature T and then one checks whether, on the basis of this solution, it is possible to find the solution of the bipartite matching problem by means of Algorithm B.

This method is not efficient since the lower is T the slower is the convergence of the Algorithm A.

Therefore it seems natural to consider a simulated annealing algorithm. The algorithm I propose and test here is the following one.

Algorithm C. Starting with $\beta = 0$, $(T = \infty)$, and $\tau_j = 0$, for any j, one iterates the following steps:

(i) one change β of a constant rate $\Delta\beta$: $\beta \to \beta + \Delta\beta$, and performs one step of Algorithm A;

(ii) one checks with Algorithm B if the solution is reached;

(iii) if not, one goes back to step (i).

Remarks.

(1) Algorithm A does not preserve $\sum_{i=1}^N \sigma_i - \sum_{j=1}^N \tau_j = 0$. This problem can be overcome by adding to the algorithm the step $\sigma_i \to \sigma_i - a$: $i = 1, \ldots, N$, $\tau_j \to \tau_j + a$: $j = 1, \ldots, N$, where $a = \frac{1}{2N}(\sum_{i=1}^N \sigma_i - \sum_{j=1}^N \tau_j)$.

(2) The solution of the cavity equations are regular functions of β (see Theorem 3.1). By taking the derivative of the solution with respect to β one gets, with a little algebra, the following differential equation.

$$\begin{aligned} \frac{d}{d\beta}\bar{\sigma}_i(\beta) &= S_i(\bar{\sigma}_1,\ldots,\bar{\sigma}_N,\bar{\tau}_1,\ldots,\bar{\tau}_N,\beta): \quad i=1,\ldots,N \\ \frac{d}{d\beta}\bar{\tau}_j(\beta) &= T_j(\bar{\sigma}_1,\ldots,\bar{\sigma}_N,\bar{\tau}_1,\ldots,\bar{\tau}_N,\beta): \quad j=1,\ldots,N \end{aligned} \tag{4.3}$$

$S_1,\ldots, S_N, T_1,\ldots, T_N$ satisfy

$$\begin{aligned} S_i + \sum_{j=1}^{N} R_{i,j} T_j &= -\frac{\bar{\sigma}_i}{\beta}: \quad i=1,..,N \\ T_j + \sum_{i=1}^{N} R_{i,j} S_i &= -\frac{\bar{\tau}_j}{\beta}: \quad j=1,..,N \end{aligned} \tag{4.4}$$

togheter with the condition

$$\sum_{i=1}^{N} S_i - \sum_{j=1}^{N} T_j = 0 \tag{4.5}$$

where $R_{i,j} = e^{-\beta(A_{i,j}-\bar{\sigma}_i-\bar{\tau}_j)}$.

Therefore a possible simulated annealing algorithm for this problem consists in solving numerically the previous equation starting with a small value of β.

Notice that, for $\Delta\beta$ small, Algorithm C essentially follows the solution of (4.3).

I have tested Algorithm C on the random matching problem studied by Mezard and Parisi.[3,4] In this case the natural scale for β is N.[3] In this test I have chosen $\Delta\beta = bN$, where $b = \frac{1}{10}$, and I have considered values of N between 10 and 200.

The average number of steps required to solve the problem seems to be less than $O(N)$, also if sometimes one needs only a few steps, and sometimes the algorithm spend an huge amount of time to find the solution. Taking into account that any step consists of $O(N^2)$ operations this algorithm seems to spend less than $O(N^3)$ operations to find the solution of the problem.

The large fluctuations in the performance of the algorithm are probably due to the fact that sometimes the problem is degenerate or quasi-degenerate. For example, in the case in which the matrix A is the null matrix, all the permutations have the same value. In this case the Algorithm B is not a good algorithm to find a solution. Nevertheless the estimate from above and from below of the cost converge to the cost and the variables $\sigma_1,\ldots, \sigma_N, \tau_1,\ldots, \tau_N$, converge to a solution of the dual problem.

Finally a comment. Algorithm B does not seems particularly good and probably its performance is comparable to that of the simplex method.

As I have said in the Introduction I think that the most interesting result of this paper is the definition of a new formalism for the bipartite matching problem based on the introduction of suitable convex functions.

I do not know if it is possible to define better algorithm to solve the bipartite matching problem based on this formalism.

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